Necessity of Visual Representations in Mathematics

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Outline

Computer Generated Visual Representations

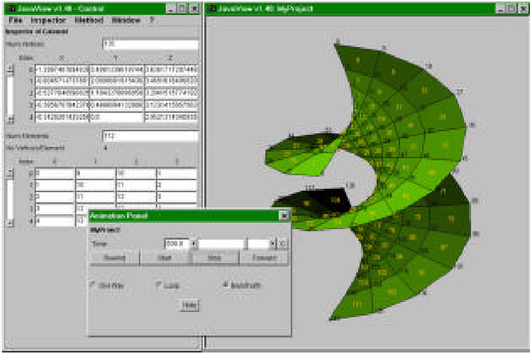


Figure : The explicit representation allows to generate exact discrete minimal surfaces without numerical errors which are especially useful for index computations. (Polthier 2002)

Epistemic Situation of Visual Representations

- "Can a visual way of acquiring a mathematical belief justify our believing it?" (Giaquinto 2007)
- A mathematical belief is knowledge, if it is true, if it does not admit "a violation of epistemic rationality in the way it is acquired and maintained", and if this belief is justified. (Giaquinto 2007)
- Beliefs that are obtained through reasoning with diagrams, namely through visual thinking, are also knowledge.

Cognitive Situation of Visual Representations.

- Even if human perception of space "violate Euclidean principles in several ways (for example, by imposing a curvature to the space) or failing to unify different scales, the way that we conceive space is not necessarily constrained by our perception.
- Following Kant's proposal, that the axioms of Euclidean geometry may constitute the most intuitive conceptualization of space not only in adults educated in the tradition of Euclidean geometry but also in cultures where this tradition is absent." (Izard et al. 2011)

Visual Representations as Formal Representation Systems.

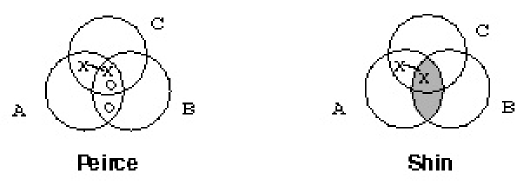


Figure : Comparison of Peirce's and Shin's Venn Diagrams. (Shin, 1996)

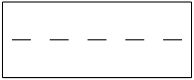
A formal representational system with diagrams can be constructed as a result of Shin's characterization of Venn diagrams with shading and Peircian existential import x.

Permuation Proofs

there are four people A, B, C, and D that must be seated in a row of five chairs. A and C must wing the empty chair. C must be closer to the center than D, who is to sit next to B. From this information we want to show:

1. The empty chair is not in the middle or on either end.
2. It is possible to tell who must be seated in the center.
3. Who the specific people are to be seated on the two ends.

The diagrammatical proof is as follows:
Let the following diagram represent five chairs:



Permuation Proofs

We have:

1. A and C must wing the empty chair.
2. C must be closer to the center than D.
3. D sits next to B.

From 1. one can split the situation in three cases since there is no difference between left and right since they mirror each other.

$\underline{A}\ \underline{\times}\ \underline{C}\ \underline{\ \ }\ \underline{\ \ }$

$\underline{\ \ }\ \underline{A}\ \underline{\times}\ \underline{C}\ \underline{\ \ }$

From 2. one can eliminate the case where C is at the end. Which leaves the cases:

$\underline{A}\ \underline{\times}\ \underline{C}\ \underline{B}\ \underline{D}$

$\underline{A}\ \underline{\times}\ \underline{C}\ \underline{D}\ \underline{B}$

Formal Proof

One of the set ups for the formal proof can be as follows:
Let $T : \{A, B, C, D, X\} \rightarrow \{-2, -1, 0, 1, 2\}$ be 1-1 and onto map such that:

i) $T(A) + 1 = T(X) = T(C) - 1$ or $T(A) - 1 = T(X) = T(C) + 1$

ii) $|T(C)| < |T(D)|$

iii) $|T(B) - T(D)| = 1$

Then show that $T(X) \neq 0$.

Proof.

Assume $T(X) = 0$ then by $i)$ either $T(A) = 1$ and $T(C) = -1$ or $T(A) = -1$ and $T(C) = 1$. Assume $T(A) = 1$ and $T(C) = -1$ then $|T(C)| = 1$ and by $ii)$ this implies $|T(D)| = 2$. Hence, either $T(D) = 2$ or $T(D) = -2$. If $T(D) = 2$ by $iii)$ $T(B) = 1$ but we have $T(A) = 1$ which contradicts with T being 1-1. If $T(D) = -2$ by $iii)$ $T(B) = -1$ but we have $T(C) = -1$ which again contradicts T being 1-1. Now assume $T(A) = -1$ and $T(C) = 1$ then $|T(C)| = 1$ and $ii)$ implies that $|T(D)| = 2$. If $T(D) = 2$ by $iii)$ $T(B) = 1$ but we have $T(C) = 1$ which contradicts with T being 1-1. If $T(D) = -2$ by $iii)$ $T(B) = -1$ but we have $T(A) = -1$ which again contradicts T being 1-1. Therefore $T(X) \neq 0$. Hence the middle chair cannot be empty. □

Formal Proof

That was only the proof of the first of the following:

1. The empty chair is not in the middle or on either end.
2. It is possible to tell who must be seated in the center.
3. Who the specific people are to be seated on the two ends.

One can prove 2. and 3. in a similar fashion by exhausting the cases with contradiction. .

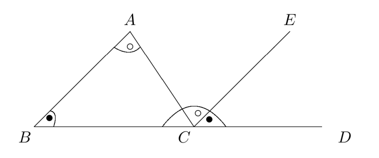
What is missing?

Traditional formal account cannot give the proof of all three cases simultaneously as the diagrammatical proof provides in this problem since there is both a model building and deduction in this problem.

Necessity of Visuals in Proofs?

Euclidean Proposition 32

The sum of interior angles of any triangle is equal to two right angles.



Proof without visuals?

Euclidean Proposition 32

Satz 20. Die Winkel eines Dreiecks machen zusammen zwei Rechte aus.

Definition. Wenn M ein beliebiger Punkt in einer Ebene a ist, so heisst die Gesamtheit aller Punkte A, für welche die Strecken MA einander congruent sind, ein Kreis; M heisst der Mittelpunkt des Kreises. Auf Grund dieser Definition folgen mit Hülfe der Axiomgruppen III IV leicht die bekannten Sätze über den Kreis, insbesondere die Möglichkeit der Konstruktion eines Kreises durch irgend drei nicht in einer Geraden gelegene Punkte sowie der Satz ber die Congruenz aller Peripheriewinkel über der näm- lichen Sehne und der Satz von den Winkeln im Kreisviereck.

Common Objections

Intuition in Mathematical Reasoning

"Because intuition turned out to be deceptive in so many instances, and because propositions that had been accounted true by intuition were repeatedly proved false by logic, mathematicians became more and more skeptical of the validity of intuition. They learned that it is unsafe to accept any mathematical proposition, much less to base any mathematical discipline on intuitive convictions. Thus a demand arose for the expulsion of intuition from mathematical reasoning, and for the complete formalization of mathematics." (Hahn 1933)

Common Objections

Intuition in Proofs

Intuition is "dispensable as a proof theoretic device; indeed, ... it has no proper place in a proof as such". (Tennant 1934)

Common Objections

Weierstrass Function

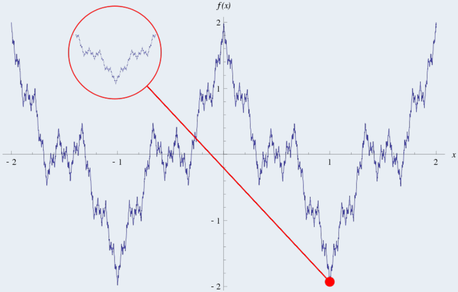


Figure : Graph of a Weierstrass function in the interval [-2, 2].

What is necessity?

Necessity in Logic

- Q is necessary for P
- P cannot be true unless Q is true
- Whenever P is true Q is true
- P cannot occur without Q

Necessity of Visual Representations

Necessity General Definition

The state or fact of being required

Mathematicians *use visual representations primarily and naturally before needing to apply to any other representation*: They *cannot avoid* visual reasoning.

Necessity of Visual Representations

We want to show:

We cannot avoid visual representations in mathematical reasoning altogether, including mental images.

Proof by Contradiction:

Assume we can avoid visual representations in mathematical reasoning altogether including mental images. As a counter example, let us take one of the beautiful and famous theorems of calculus, the intermediate value theorem. It states the following:

Necessity of Visual Representations

The Intermediate Value Theorem:

If f is a function which is continuous at every point of the interval $[a, b]$ and $f(a) < 0$, $f(b) > 0$ then $f(x) = 0$ at some point $x \in (a, b)$.
Intuitively, the naive definition of continuity which says that the graph of a continuous function has no gaps can be used to explain the fact that a function which starts on below the x-axis and finishes above it must cross the axis.

Necessity of Visual Representations

Proof:

The idea of the proof is to look for the first point at which the graph of f crosses the axis. Let $X = \{x \in [a, b] \mid f(y) \leq 0 \text{ for all } y \in [a, x]\}$. Then X is non-empty since $a \in X$, and $X \subset [a, b]$ so it is bounded. Hence, by the Completeness Axiom, X has a least upper bound α (say). We claim that $f(\alpha) = 0$.

Proof of the claim: We will show that either of the assumptions $f(\alpha) > 0$ or $f(\alpha) < 0$ leads to a contradiction and the result then follows from the Trichotomy property of the Order Axiom. So suppose $f(\alpha) > 0$. Say $f(\alpha) = \epsilon$. Then for some $\delta > 0$ we have $f(x) > 0$ for x lying in the interval $(\alpha - \delta, \alpha + \delta)$. But then $\alpha - \delta$ would be an upper bound of X, contradicting the fact that α is the least upper bound. Similarly, suppose $f(\alpha) < 0$. Say $f(\alpha) = -\epsilon$. Then for some $\delta > 0$ we have $f(x) < 0$ for x lying in the interval $(\alpha - \delta, \alpha + \delta)$. But this is a contradiction since α is an upper bound of X. This completes the proof.

Necessity of Visual Representations

IVT Graph

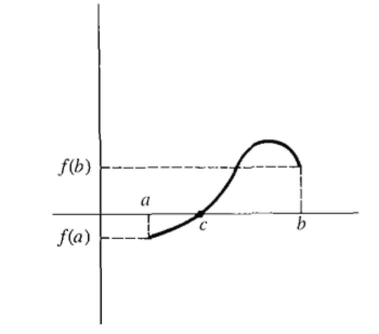


Figure : Graph of a function satisfying the assumptions of the Intermediate Value Theorem.

Conclusion

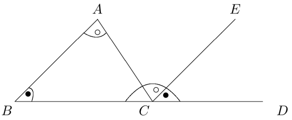


Figure : Drawing used in the Euclidean proof.

```
\setlength{\unitlength}{1cm}
\begin{picture}(10,5)
\put(1,1){\line(1,1){3}}
\put(4,4){\line(2,-3){2}}
\put(6,1){\line(1,1){3}}
\put(1,1){\line(1,0){9}}
\qbezier(1.5,1.5)(1.75,1.45)(1.6,1)
\put(1.35,1.15){$\bullet$}
%\qbezier(6.5,1.5)(6.75,1.45)(6.6,1)
\put(6.35,1.1){$\bullet$}

\qbezier(3.5,3.5)(4,3.20)(4.3,3.5)
\put(3.87,3.40){$\circ$}

%\qbezier(6.5,1.5)(6,1.9)(5.7,1.5)
\put(6,1.3){$\circ$}

\qbezier(5,1)(6,2.3)(7,1) %half circle

\put(0.5,0.5){$B$}
\put(3.8,4.2){$A$}
\put(5.5,0.5){$C$}
\put(10.5,0.5){$D$}
\put(8.8,4.2){$E$}

\end{picture}
```

Figure : LaTeX code needed to draw Fig. 5.

Conclusion

Intuitions

- Dehumanization attempts of mathematics does not make sense
- Reasons behind such motivations are unfounded
- Weierstrass function could have been represented visually if the fractals were known earlier

Conclusion

Intuitions, Visual Representations, Mathematics

- Visualization is a natural part of mathematics and can be used in the context of justification
- There are actually uses of visual representations even in purely formal proofs
- Eliminating external visual representations does not eradicate the visualization
- Visual representations are necessarily used in mathematics, in discovery and in proofs.