Kantian Characterization of Mathematics

Özge Ekin Gün, PhD.

Freie University of Berlin
Insitute of Philosophy

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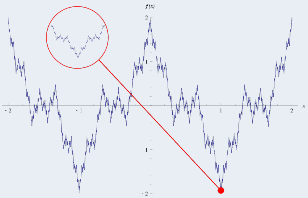
Outline

Introduction

Background and Motivation

Dehumanization of mathematics and elimination of intuitions

- Discovery of geometrical or topological monsters → demonstrating unreliability of intuition
  Example: Weierstrass' everywhere continuous but nowhere differentiable function



- Developments in modern logic → formalization of mathematics → insistence on the usage of only sentential formal representations

Introduction

Importance

Visual reasoning in mathematics with respect to Kantian characterization of mathematics

- Objective background for visual reasoning in mathematics
- Visual representations are not only psychological tools or have only heuristic usage
- Validity of Kant's characterization of mathematics as synthetic a priori

Introduction

Problem

- Many interpretations of Kantian characterization of mathematics
- Many approaches to visual reasoning in mathematics: computer generated images, epistemological, cognitive and formal

How to make sense of it all?

- Meta-analysis
- Natural link between visualization and Kant's synthetic a priori

Kant's Synthetic A priori

Synthetic a priori as a method

- Not primarily truth statements in propositional forms
- Synthetic a priori method in context of justification

Visualization

- External visual representations can be formalized but...
- Visualization remains (exhibiting intuition a priori in pure intuition space)

Kant's Synthetic A priori

Kantian intuitions and visual representations

Visual representations are intuitions (Anschauungen)

Not only:

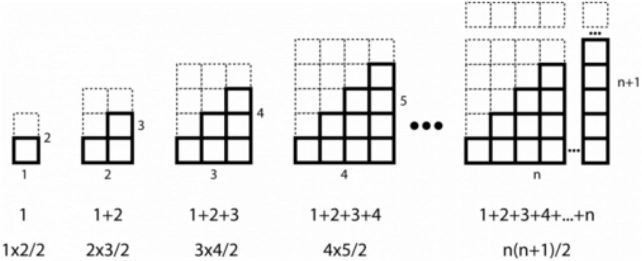
- *a* triangle used in an Euclidean proof, drawn on a medium or visualized in mental space
- Venn diagrams

But also:

- ellipsis (...) in 1+2+3+...
- a letter symbol such as "a" representing a general property in $a = a$ (law of identity)

1+2+3+4+... Proof by Picture

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$



1+2+3+4+... Proof by Induction

Let the property $P(n)$ be the equation $1 + 2 + \cdots + n = n(n+1)/2$. Show that the property $P(n)$ is true for $n = 1$. We must show that $1 = \frac{1(1+1)}{2}$. The right hand side of the equation is $\frac{1(1+1)}{2} = \frac{2}{2} = 1$, which is the same as the left hand side. So the property is true for $n = 1$. Show that for all integers $n = k$, if $P(k)$ is true, then so is $P(k+1)$. For the induction hypothesis, suppose $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$, for some integer $k \geq 1$. From this we must show that $1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}$. The left hand side of the equation can be expanded to: $(1 + 2 + \cdots + k) + (k+1)$. Substituting using the induction hypothesis, this is: $\frac{k(k+1)}{2} + (k+1)$. Finding a common denominator and simplifying, we have:

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{(k+1)\cdot 2}{2} \qquad (1)$$

$$= \frac{(k+1)(k+2)}{2} \qquad (2)$$

which is what we were trying to show. QED.

1+2+3+4+... Gauss Way

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \dots + (n-2) + (n-1) + n$$

$$\sum_{i=1}^{n} i = n + (n-1) + (n-2) + \dots + 3 + 2 + 1$$

$$2 * \sum_{i=1}^{n} i = (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1) + (n+1)$$

$$= n(n+1)$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Conclusion

Intuitions, Visual Representations, Mathematics

- There is a process that helps one to look at the general complicated representation in a simplified particular way
- Any representation allowing this process is a visual representation or intuition in Kantian sense

Conclusion

Mathematical truths and methods

- Although mathematics has infallible mathematical truths immune to empirical overthrown, it is practiced by human beings
- Formal sentential method is not the only way to reach mathematical truths
- Mathematical objects and methods evolve and are revisable

Conclusion

Intuitions

- Kant's characterization of mathematics can provide an objective background for visual reasoning in mathematics.
- Visual representations are not only psychological tools or have only heuristic usage.
- There is usage of synthetic a priori method in the context of justification in mathematics.
- Visual representations and intuition are indispensable in mathematics.

Outlook in Future Research

Deciding what is valid to use in mathematics affects how it is taught and how published media appears

A Klein bottle may be parametrized by the following equations:

$$x = \begin{cases} a\cos(u)\,(1+\sin(u)) + r\cos(u)\cos(v) & 0 \le u < \pi \\ a\cos(u)\,(1+\sin(u)) + r\cos(v+\pi) & \pi < u \le 2\pi \end{cases}$$

$$y = \begin{cases} b\sin(u) + r\sin(u)\cos(v) & 0 \le u < \pi \\ b\sin(u) & \pi < u \le 2\pi \end{cases}$$

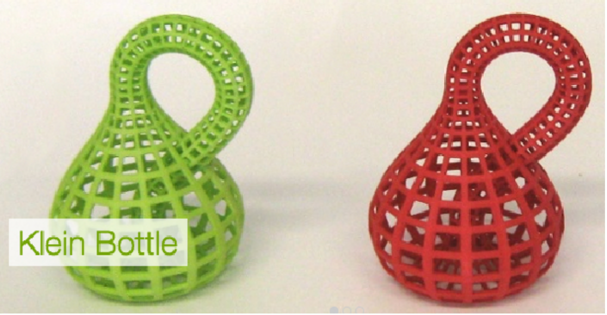
$$z = r\sin(v)$$

where $v \in [0, 2\pi]$, $u \in [0, 2\pi]$, $r = c\left(1 - \frac{\cos(u)}{2}\right)$ and a, b, c are chosen arbitrarily.

http://planetmath.org/kleinbottle

Outlook in Future Research

Demanding the usage of only sentential formal methods in such visual era is incomprehensible



https://www.vismath.eu/

Outlook in Future Research

Theory of Visual Representations:

- Visual thinking as a form of valid reasoning in mathematics
- Visualization can be learned and improved
- Theory of visual representations as a branch of mathematics